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# A new three-point approximation approach for design optimization problems

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SUMMARY

In this paper, a newly developed three-point approximation scheme is proposed. The expression of this scheme consists of a linear combination of the direct and reciprocal linear Taylor expansions as well as of the lumped diagonal terms of the second-order direct and inverse terms. The unknown parameters of the expression are computed on the basis of the function and gradient values at three points in the design space. Based on this approach, the accuracy of the existing constraint approximation methods can be improved. The effectiveness of the proposed approach is demonstrated on a number of numerical examples. The numerical results are also compared with those of the previously published work. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: multi-point approximations; optimization; design

## 1. INTRODUCTION

In order to reduce the computational efforts and to enhance the efficiency of the optimization process, the use of explicit albeit approximate expression for the constraint and objective functions in terms of design variables is a common technique in any structural optimization scheme. It has been widely known that more accurate function approximations can reduce the analysis cost greatly. Therefore, since the introduction of approximation concepts by Schmit *et al.* [1] in the mid-1970s, the development of high-quality and reliable constraint approximation scheme has been the subject of many research activities. A lot of attempts have been made to approximate the constraint and objective functions in multidisciplinary optimization problems which are usually computationally expensive to evaluate with less computational efforts and high accuracy. As pointed out by Barthelemy and Haftka [2], there are three categories of approximations according to their range of applicability in the design space. They are named as *local*, *global* and *mid-range approximation*, respectively.

The *local approximation* is constructed by using the local information of function values and their sensitivities at single design point. This approximation is only valid at the vicinity of the expanding point. The simplest approach of this type is the linear approximation based on the

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Taylor series. Another often used approach of this type is the reciprocal approximation. It is in nature the Taylor series expansion in terms of reciprocal variables. Starner and Haftka [3] have also shown that a hybrid constraints mixing the direct variables and reciprocal variables can yield a more conservative approximation, which has the advantage of giving convex sub-problem.

The *global approximation*, on the other hand, tries to construct approximate functions, usually polynomials, for whole area of the design parameters. Global approximation approaches are often used to modify the formulation of the original design problems, in most case of which the explicit formulation is not known, and to generate an alternative formulation that is more tractable. The most common global approximation method is the *response surface approach*, in which the function is sampled at a number of design points, and then an analytical expression called the response surface polynomial is fitted to the sample data. Construction of response surface often relies heavily on the theory of design-of-experiment. The linear or non-linear regression technique is used to fit the surface, and the fitted response surface is employed to search an optimum design. The problem associated with global approximation is that it requires many response analyses like finite element method at many design points in the design space for obtaining the approximate function which is valid for whole area of the design parameters. Therefore, the global approximation has been believed usable only for design problems with a few design parameters, only when the sensitivity information is not available or costly.

*Mid-range approximation* is an attempt to endow local function approximation with a wider range of applicability. The use of information at several points can achieve this purpose. A two-point approximation was proposed by Fadel *et al.* [4] to enhance the quality of approximation. Wang and Grandhi [5–7] generalized the work of Fadel and proposed a series of new multi-point approximation approaches by using two or more points information of optimization iterations. In their work, intervening variables have been used to control the non-linearity of the approximations. Good results have been reported by employing this approach to solve the size and configuration optimization problems. Salajegheh [8] developed a three-point approximation scheme based on the function and gradient information at three different design points. He also applied this method to optimize the plate structures subjected to stress and frequency constraints and obtained satisfactory results. Sui [9] has also proposed a two-point approximation approach by employing the information obtained at two design points. The state art of multi-point approximation has been reviewed intensively by Wang and Grandhi [5].

In the present study, a newly developed three-point approximation scheme (TPA) is proposed. The expression of this scheme consists of a linear combination of the direct and reciprocal linear Taylor expansions as well as of the lumped diagonal terms of the second-order direct and inverse terms. The unknown parameters of the expression are computed on the basis of the function and gradient values at three points in the design space. Based on this approach, the accuracy of the existing constraint approximation methods can be improved. The effectiveness of the proposed approach is demonstrated on a number of numerical examples. The numerical results are also compared with those of the previously published works.

## 2. REVIEW OF MULTI-POINT APPROXIMATIONS

In this section, a brief review of the earlier developments of multi-point approximation is presented to understand well the previously proposed method. In the following, $\mathbf{X} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ is defined as a vector of design variables. $\mathbf{X}^i = (x_1^i, x_2^i, \ldots, x_n^i)^{\mathrm{T}}$ refers to the $i$th data point in the design space.

### 2.1. *Two-point approximation approach*

*2.1.1. Two-point exponential approximation approach.* This approximation scheme has been introduced by Fadel *et al.* [4]. It is a linear Taylor approximation in terms of the following intermediate variables:

$$y_i = x_i^{p_i}, \quad i = 1, 2, \ldots, n \tag{1}$$

where the exponent $p_i$ for each design variable is determined by matching the derivatives of the approximate function with those of the exact function at previous data point.

*2.1.2. Two-point adaptive non-linear approximation approach-TANA.* This approach has been proposed by Wang and Grandhi [6]. In this approach, the non-linear index of each design variable was chosen as the same value and calculated numerically by matching the approximate and exact function values at two different points.

*2.1.3. TANA-1 and TANA-2 approximation approach.* To utilize more information in constructing a better approximation, Wang and Grandhi [5] have proposed two kinds of improved two-point approximation schemes by employing both function and derivative values of two points. In TANA-1 approach, the approximated function is expanded at one point $\mathbf{X}^1$ as

$$f^*(\mathbf{X}) = f(\mathbf{X}^1) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^1)}{\partial x_i} \frac{(x_i^1)^{1-p_i}}{p_i} \left[x_i^{p_i} - (x_i^1)^{p_i}\right] + \varepsilon \tag{2}$$

where $\varepsilon$ is a constant, representing the residue of the first-order Taylor approximation in terms of the intervening variables $y_i = x_i^{p_i}$. To evaluate $p_i$ and $\varepsilon$, the approximate function value and its derivatives are matched with those of the exact function at another point $\mathbf{X}^2$.

In TANA-2 approach, the approximate function is obtained by expanding the function at the point $\mathbf{X}^2$ and replacing the constant term $\varepsilon$ in TANA-1 by the term $\varepsilon_2 \sum_{i=1}^{n} \left[x_i^{p_i} - (x_i^2)^{p_i}\right]^2/2$, which takes the effect of second-order terms of Taylor expansion into consideration. It can be written as follows:

$$f^*(\mathbf{X}) = f(\mathbf{X}^2) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^2)}{\partial x_i} \frac{(x_i^2)^{1-p_i}}{p_i} \left[x_i^{p_i} - (x_i^2)^{p_i}\right] + \varepsilon_2 \sum_{i=1}^{n} \left[x_i^{p_i} - (x_i^2)^{p_i}\right]^2/2 \tag{3}$$

In this approach, the values of $p_i$ and $\varepsilon_2$ have been evaluated by matching the derivatives and the function values with those of the exact function at previous point $\mathbf{X}^1$.

It has been reported that good results can be obtained by employing these methods for constraint approximations. They can provide better accuracy than linear and reciprocal methods for highly nonlinear problems. But it should be noted that since the equations for the determination of the exponent $p_i$ are non-linear, numerical iterative search process is required for the evaluation of these quantities.

*2.1.4. Sui's two-point approximation approach [9].* In this approach, the approximate function is expressed as

$$f^*(\mathbf{X}) = \alpha + \sum_{i=1}^{n} \beta_i x_i + \sum_{i=1}^{n} \gamma_i / x_i \tag{4}$$

The $2n+1$ unknown parameters $\alpha_i, \beta_i$ and $\gamma$ are computed by matching the approximate function value and exact function value at one matching point, and the derivatives of the approximate function and exact function at both of the expanding and matching point. Good results are reported by employing this approach for the solutions of structural optimization problems.

### 2.2. *Three-point approximation approach*

Recently, Salajegheh [8] has proposed a three-point approximation approach to achieve a higher quality approximation. This approach can be described as follows. Let $\mathbf{X}^1$, $\mathbf{X}^2$ and $\mathbf{X}^3$ be three consecutive points in the design space. By choosing the intermediate variables as $y_i = x_i^{p_i}$ and expanding the function at the middle-point $\mathbf{X}^2$ in terms of $y_i$, the approximate function can be expressed as

$$f^*(\mathbf{X}) = f(\mathbf{X}^2) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^2)}{\partial x_i} \frac{(x_i^2)^{1-p_i}}{p_i} \left[x_i^{p_i} - (x_i^2)^{p_i}\right] + \sum_{i=1}^{n} c_i \left[x_i^{p_i} - (x_i^2)^{p_i}\right]^2 /2 + \varepsilon \tag{5}$$

To evaluate the unknown parameters $p_i$, $c_i$ and $\varepsilon$, $2n+1$ equations in total are required. Here, $2n$ equations are obtained by differentiating $f^*(\mathbf{X})$ and matching the derivatives with those of the exact function at points $\mathbf{X}^1$ and $\mathbf{X}^3$. Another equation is obtained by matching the exact and approximate function values at one of the point $\mathbf{X}^1$ or $\mathbf{X}^3$. It has been reported by the author that efficient optimum design of plate structures with stress, displacement and frequency constraints can be achieved by employing this approach. Just like that of TANA-2, however, the main drawback of this approach is that a set of coupled non-linear equations need to be solved for the evaluation of the values of $p_i$, which is usually high in computational cost for most of the practical design optimization problems. When the numbers of the design variables and active constraints are large, the determinations of the values of $p_i$ may even take much more floating operations than that of FEM analysis. Moreover, because of the non-linear characteristics of the equations, we encounter the difficulty that the values of $p_i$ cannot always be available for arbitrary function and gradient values of the expanding and matching points.

### 2.3. *Multi-point approximation approach*

Wang and Grandhi [7] proposed an approximation approach based on the multiple function and gradient information by using Hermite interpolation concepts. This approximation possesses the property of reproducing the function and gradient information of known data points. But it has been reported that this approach is not as stable as TANA-1 and TANA-2 approach when used for the solutions of structural optimization problems.

## 3. NEW THREE-POINT APPROXIMATION SCHEME

In this paper, a new three-point approximation approach (TPA) is proposed. It is based on the linear combination of Taylor expansions in terms of both direct and reciprocal variables. Let $\mathbf{X}^1$, $\mathbf{X}^0$ and $\mathbf{X}^2$ be three consecutive points. The approximate function is assumed as taking the following form:

$$f^*(\mathbf{X}) = f(\mathbf{X}^0) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{X}^0)}{\partial x_i}\left[\alpha_i(x_i - x_i^0) + \beta_i\left(x_i^0 - \frac{(x_i^0)^2}{x_i}\right)\right]$$

$$+ \frac{1}{2}c_1\sum_{i=1}^{n}(x_i - x_i^0)^2 + \frac{1}{2}c_2\sum_{i=1}^{n}\left(\frac{1}{x_i} - \frac{1}{x_i^0}\right)^2 \tag{6}$$

From Equation (6) it can be clearly seen that this approximation scheme is just a linear combination of first-order Taylor expansions in terms of both direct and reciprocal variables and is also taken the second-order effect into consideration by assuming the Hessian matrix has only diagonal elements of the same value ($c_1$ for direct variables, and $c_2$ for reciprocal variables).

Denoting the current design point as $\mathbf{X}^0$ and the other two design points as $\mathbf{X}^1$ and $\mathbf{X}^2$, respectively, the matching conditions for the evaluation of the unknown parameters $\alpha_i, \beta_i$, $(i = 1, 2, \ldots, n)$, and $c_1$, $c_2$ can be written as follows:

$$\nabla f^*(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^1} = \nabla f(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^1},\ \nabla f^*(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^2} = \nabla f(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^2},\ f^*(\mathbf{X}^1) = f(\mathbf{X}^1),\ f^*(\mathbf{X}^2) = f(\mathbf{X}^2) \tag{7}$$

where $f(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^i}$ and $\nabla f(\mathbf{X})|_{\mathbf{X}=\mathbf{X}^i} = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)^{\mathrm{T}}|_{\mathbf{X}=\mathbf{X}^i}$ represent the exact value of the constraint function and its derivative at data point $\mathbf{X}^i$, respectively.

Since Equation (7) gives $2n+2$ linear algebra equations, then the $2n+2$ unknowns $\alpha_i, \beta_i$ and $c_1$, $c_2$ can be evaluated by solving this system of linear algebra equations. In the following derivation, we assume that the function and gradient information have already been obtained at the point of $\mathbf{X}^0$, $\mathbf{X}^1$ and $\mathbf{X}^2$.

From matching conditions, we have

$$\frac{\partial f}{\partial x_i}(\mathbf{X}^0)\left[\alpha_i + \beta_i\left(\frac{x_i^0}{x_i^1}\right)^2\right] + c_1\left(x_i^1 - x_i^0\right) + c_2\left(\frac{-1}{(x_i^1)^2}\right)\left(\frac{1}{x_i^1} - \frac{1}{x_i^0}\right) = \frac{\partial f}{\partial x_i}(\mathbf{X}^1) \tag{8}$$

$$\frac{\partial f}{\partial x_i}(\mathbf{X}^0)\left[\alpha_i + \beta_i\left(\frac{x_i^0}{x_i^2}\right)^2\right] + c_1\left(x_i^2 - x_i^0\right) + c_2\left(\frac{-1}{(x_i^2)^2}\right)\left(\frac{1}{x_i^2} - \frac{1}{x_i^0}\right) = \frac{\partial f}{\partial x_i}(\mathbf{X}^2) \tag{9}$$

$$f^*(\mathbf{X}^1) = f(\mathbf{X}^0) + \sum_{i=1}^{n}\left(a_i^1\alpha_i + b_i^1\beta_i\right) + c_1 w^1 + c_2\theta^1 = f(\mathbf{X}^1) \tag{10}$$

$$f^*(\mathbf{X}^2) = f(\mathbf{X}^0) + \sum_{i=1}^{n}\left(a_i^2\alpha_i + b_i^2\beta_i\right) + c_1 w^2 + c_2\theta^2 = f(\mathbf{X}^2) \tag{11}$$

here

$$w^1 = \frac{1}{2}\sum_{i=1}^{n}(x_i^1 - x_i^0)^2 \quad w^2 = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_i^0)^2 \tag{12}$$

$$\theta^1 = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{x_i^1} - \frac{1}{x_i^0}\right)^2 \quad \theta^2 = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{x_i^2} - \frac{1}{x_i^0}\right)^2 \tag{13}$$

Let

$$e_i^1 = \left(\frac{-1}{(x_i^1)^2}\right)\left(\frac{1}{x_i^1} - \frac{1}{x_i^0}\right) \quad e_i^2 = \left(\frac{-1}{(x_i^2)^2}\right)\left(\frac{1}{x_i^2} - \frac{1}{x_i^0}\right) \tag{14}$$

$$f_i = \left(\frac{x_i^0}{x_i^1}\right)^2 - \left(\frac{x_i^0}{x_i^2}\right)^2 \quad \mu_i = \left(\frac{\partial f}{\partial x_i}(\mathbf{X}^1) - \frac{\partial f}{\partial x_i}(\mathbf{X}^2)\right)\Big/ f_i \tag{15}$$

$$y^\gamma = \sum_{i=1}^{n}\left\{\mu_i\left[x_i^0 - (x_i^0)^2/x_i^\gamma\right] + (x_i^\gamma - x_i^0)\left(x_i^0/x_i^\gamma\right)^2 + \frac{\partial f}{\partial x_i}(\mathbf{X}^\gamma)(x_i^\gamma - x_i^0)\right\}, \quad \gamma = 1,2 \tag{16}$$

$$z^\gamma = w^\gamma + \sum_{i=1}^{n}\left\{\left[x_i^0 - (x_i^0)^2/x_i^\gamma\right](x_i^2 - x_i^1)/f_i + (x_i^\gamma - x_i^0)(x_i^2 - x_i^1)^2/f_i + (x_i^\gamma - x_i^0)^2\right\}, \quad \gamma = 1,2 \tag{17}$$

$$s^\gamma = \theta^\gamma + \sum_{i=1}^{n}\left\{\left[x_i^0 - (x_i^0)^2/x_i^\gamma\right](e_i^2 - e_i^1)/f_i + (x_i^\gamma - x_i^0)(e_i^2 - e_i^1)\left(x_i^0/x_i^1\right)^2/f_i + (x_i^\gamma - x_i^0)\,e_i^\gamma\right\}, \quad \gamma = 1,2 \tag{18}$$

By solving the equations in (8)–(11), we can obtain that

$$c_1 = \frac{[f(\mathbf{X}^1) - f(\mathbf{X}^0) - y^1]s^2 - [f(\mathbf{X}^2) - f(\mathbf{X}^0) - y^2]s^1}{z^1 s^2 - z^2 s^1} \tag{19}$$

$$c_2 = \frac{f(\mathbf{X}^1) - f(\mathbf{X}^0) - y^1 - c_1 z^1}{s^1} \tag{20}$$

Once after the values of $c_1$ and $c_2$ are obtained, the values of $\alpha_i, \beta_i$ can be computed by inserting the values of $c_1$ and $c_2$ into Equations (8) and (9). Up to now, all the values of the unknown coefficients have been determined by employing the matching conditions.

It should be noted that for the new approach proposed here, all of the unknown coefficients can be identified in a closed form, no numerical search process is required unlike in TANA-1, TANA-2 and Salajegheh's method. Therefore, much computational efforts can be reduced for the solution of unknown coefficients.

 

## 4. NUMERICAL EXAMPLES

In this section, several numerical examples are selected to demonstrate the effectiveness and the accuracy of the proposed approximation scheme, and the results are also compared with those obtained by other methods such as TANA-1, linear and reciprocal approximation approach. This section includes two parts. The first part is for simple problems with closed-form solutions, whereas the second part presents applications to structural optimization.

### 4.1. *Application to polynomial approximation*

In all of the examples, the test points are generated using

$$\mathbf{X} = \mathbf{X}^0 + \alpha\mathbf{D} \tag{21}$$

where $\mathbf{X}^0$ is an expanding point, $\alpha$ is a step length and $\mathbf{D}$ is a vector representing the search direction. We use the relative error percentage and the absolute error in evaluating the accuracy. For all of the test examples, the relative error index is defined as

$$\text{Relative error percentage} = \frac{\text{Approximation} - \text{Exact}}{\text{Exact}} \times 100 \text{ per cent} \tag{22}$$

and the absolute error index is defined as

$$\text{Absolute error} = \text{Exact} - \text{Approximation} \tag{23}$$

*Example 1*. This example is taken from Reference [5]. The function to be approximated is defined as

$$f(x) = \frac{10}{x_1} + \frac{30}{x_1^3} + \frac{15}{x_2} + \frac{2}{x_2^3} + \frac{25}{x_3} + \frac{108}{x_3^3} + \frac{40}{x_4} + \frac{47}{x_4^3} - 1.0 \tag{24}$$

This example has been examined by employing the linear, reciprocal, TANA-1, and the proposed TPA scheme with the selections of $\mathbf{D}$ as $\mathbf{D}_1 = (1,1,1,1)^T$, $\mathbf{D}_2 = (0,1,0,1)^T$, $\mathbf{D}_3 = (1,-1,1,-1)^T$ and $\mathbf{D}_4 = (1,0,1,0)^T$, respectively. For this example, the expanding points for all methods are selected as $\mathbf{X}^0 = (1.2, 1.2, 1.2, 1.2)^T$. The matching point for TANA-1 is selected as $\mathbf{X}^1 = (1.0, 1.0, 1.0, 1.0)^T$ as has been done in Reference [5]. According to Reference [5], the exponents $p_i$ for $x_1, x_2, x_3$ and $x_4$ are $-2.7625$, $-1.5$, $-2.825$ and $-2.4785$, respectively, and the residual constant $\varepsilon$ is $-0.0862$. $\mathbf{X}^1$ is also used as one of the matching points for TPA. Another matching points required for TPA are selected as $\mathbf{X}^2 = (0.8, 0.8, 0.8, 0.8)^T$ for $\mathbf{D}_1$, and $\mathbf{X}^2 = (1.3, 1.3, 1.3, 1.3)^T$ for $\mathbf{D}_2$, $\mathbf{D}_3$ and $\mathbf{D}_4$, respectively.

The relative errors of various methods are plotted in Figures 1(a)–1(d) for the four test cases. These figures show that for this example, TANA-1 approach has the best accuracy among all methods compared. This is due to the fact that the function to be approximated in this example is almost separable and the form of which is almost the same as that of the assumed function in TANA-1. The proposed TPA scheme also behaves well for this problem (with the maximum relative error less than 11 per cent in case 1, 5 per cent in case 2, 7.17 per cent in case 3 and 7.9 per cent in case 4), and has almost the same accuracy as that of TANA-1 for all test points. The accuracy of TPA is much better than that of linear and reciprocal approximations especially when the value of step length is large.

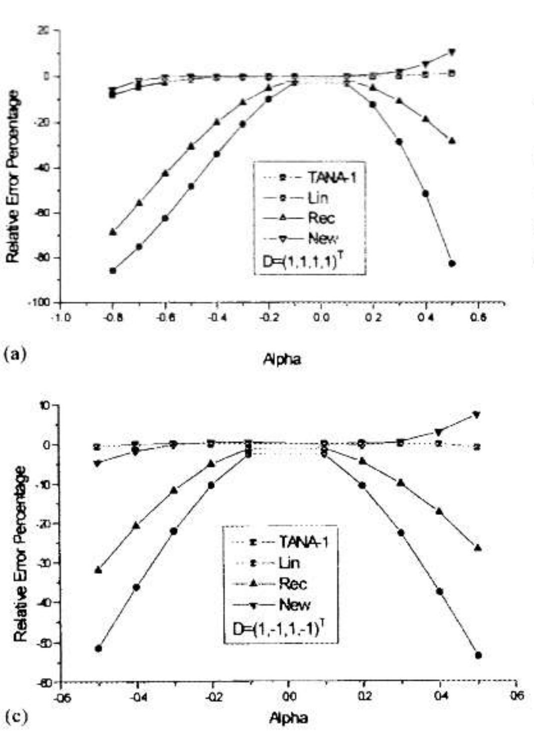


Figure 1. Example 1: (a) case 1; (b) case 2; (c) case 3; (d) case 4.

*Example 2*. This example is also taken from Reference [5]. The function to be approximated is defined as

$$f(x) = 180x_1 + 20x_2 - 3.1x_3 + 0.24x_4 - 5x_1x_2 + 37x_1x_3 + 8.7x_2x_4 - 3x_3x_4 - 0.1x_1^2 + 0.001x_2^2x_3 + 95x_1x_4^2 - 81x_4x_3^2 + x_1^3 - 6.2x_2^3 + 0.48x_3^3 + 22x_4^3 - 1.0 \tag{25}$$

This example has been examined by employing the linear, reciprocal, TANA-1, and the proposed TPA scheme, respectively, with the selections of $\mathbf{D}$ as $\mathbf{D}_1 = (1,0,1,0)^T$, $\mathbf{D}_2 = (0,1,0,1)^T$ and $\mathbf{D}_3 = (1,-1,1,-1)^T$, respectively. For this example, the expanding points for all methods are selected as $\mathbf{X}^0 = (0.8, 0.8, 0.8, 0.8)^T$. The matching point for TANA-1 is selected as $\mathbf{X}^1 = (1.0, 1.0, 1.0, 1.0)^T$ as has been done in Reference [5]. Under this circumstance, the exponent indices $p_i$ for TANA-1 are $1.6, -2.5255, 3.2375$ and $2.9625$ for $x_1, x_2, x_3$ and $x_4$ [5]. The residual constant for TANA-1 is $\varepsilon = -0.1183$. The matching points of TPA for $\mathbf{D}_1$ are $\mathbf{X}^1 = (1.2, 1.2, 1.2, 1.2)^T$ and $\mathbf{X}^2 = (0.5, 0.5, 0.5, 0.5)^T$, respectively. The matching points of TPA for $\mathbf{D}_2$ and $\mathbf{D}_3$ are $\mathbf{X}^1 = (1.0, 1.0, 1.0, 1.0)^T$ and $\mathbf{X}^2 = (0.7, 0.7, 0.7, 0.7)^T$, respectively.

The relative errors of various methods with different choices of the direction vector $\mathbf{D}$ are plotted in Figures 2(a)–2(d), respectively. For this highly non-linear example, all figures show that the present TPA approach has higher accuracy in comparison with other methods and improves the

 

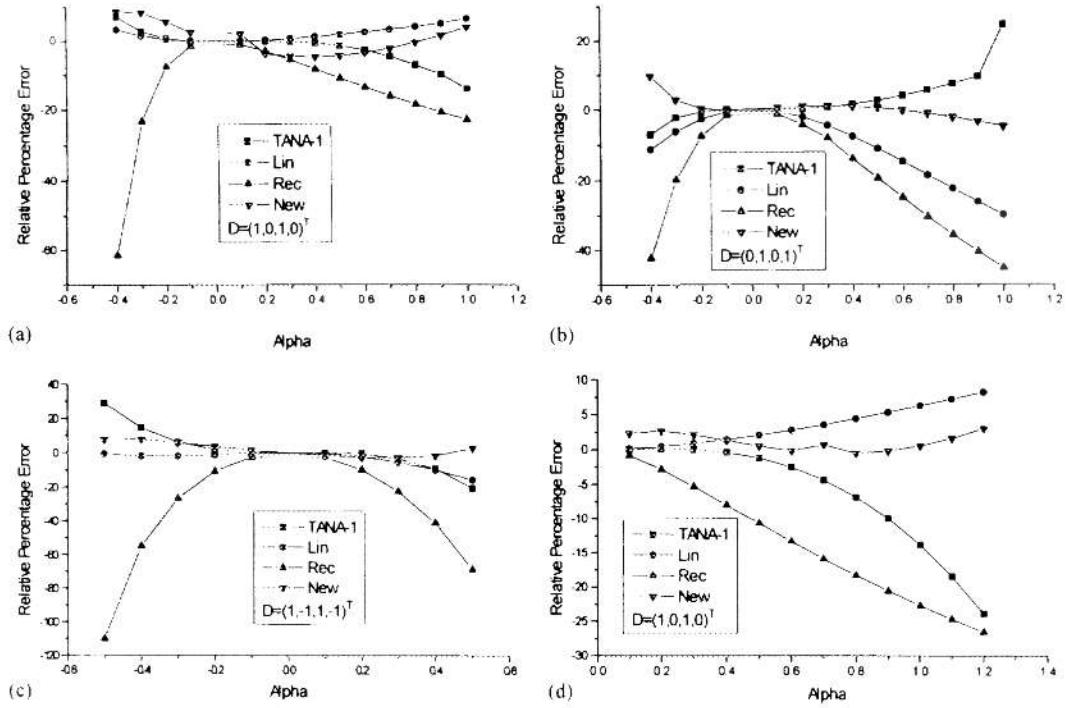


Figure 2. Example 2: (a) case 1 (1); (b) case 2; (c) case 3; (d) case 1 (2).

performance of linear and reciprocal approximations substantially. For the first two test cases, TANA-1 behaves well when $0 \leqslant \alpha \leqslant 0.5$, but when $\alpha > 0.5$, the relative error becomes large. When $\alpha = 1.0$, TANA-1 has the maximum error $-13.81$ per cent for $\mathbf{D}_1$, and 24.54 per cent for $\mathbf{D}_2$, respectively. On the other hand, the present TPA scheme performs well even when the test point is far from the expanding point. When $\alpha = 1.0$, its relative error is only 4.25 per cent for $\mathbf{D}_1$, and $-4.88$ per cent for $\mathbf{D}_2$, respectively. This can be attributed to the fact that TPA has utilized more information of the original function than the two-point approximation schemes. Thus, the trust region of the approximation function is extended. For case 3 of this problem, the results show that TPA also behaves well when the design variables are changed along alternative opposite directions. The maximum relative error of TPA is only 7.81 per cent when $-0.5 \leqslant \alpha \leqslant 0.5$, whereas the TANA-1, linear and reciprocal approximation approach have the maximum error of 28.8, $-110$ and $-15.33$ per cent, respectively.

Figure 2(d) shows the relative errors of different methods for $\mathbf{D}_1$ when $\alpha$ only takes the positive value. For this case, the matching points for TPA are selected as $\mathbf{X}^1 = (1.0, 1.0, 1.0, 1.0)^{\mathrm{T}}$ and $\mathbf{X}^2 = (1.4, 1.4, 1.4, 1.4)^{\mathrm{T}}$, respectively. The expanding and matching points for TANA-1 are the same as those in the above two cases.

Figure 2(d) indicates that if the matching points of TPA are selected in the test direction, the performance will be improved more. For this case, the relative error of TPA is only 3.07 per cent when $\alpha = 1.2$, whereas the TANA-1, linear and reciprocal approximation approach have the error of $-23.86$, 8.29 and $-26.55$ per cent, respectively, when $\alpha = 1.2$.

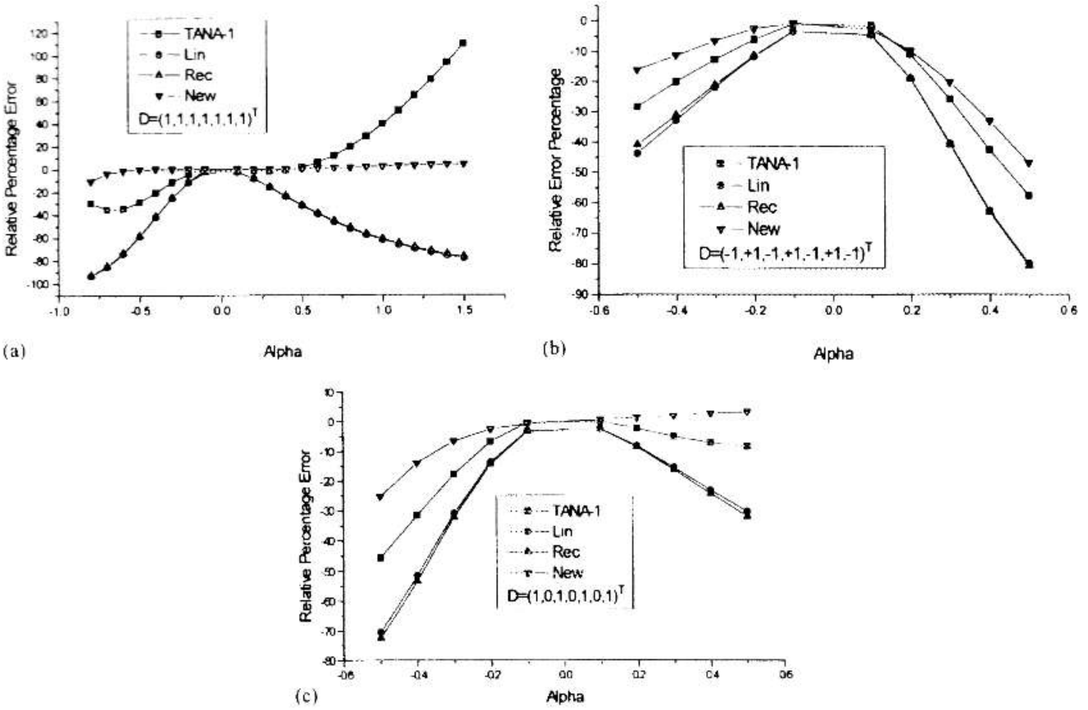


Figure 3. Example 3: (a) case 1; (b) case 2; (c) case 3.

*Example 3.* This example which is also taken from Reference [5] is highly complex and non-linear. The function to be approximated is defined as

$$f(x) = 10x_1x_2^{-1}x_4^2x_6^{-3}x_7^{0.125} + 15x_1^{-1}x_2^{-2}x_3x_4x_5^{-1}x_7^{-0.5} + 20x_1^{-2}x_2x_4^{-1}x_5^{-2}x_6 + 25x_1^2x_2^2x_3^{-1}x_5^{0.5}x_6^{-2}x_7 \tag{26}$$

This example has been examined by employing the linear, reciprocal, TANA-1, and the proposed TPA scheme with the selections of $\mathbf{D}$ as $\mathbf{D}_1 = (1,1,1,1,1,1,1)^T$, $\mathbf{D}_2 = (-1,1,-1,1,-1,1,-1)^T$ and $\mathbf{D}_3 = (1,0,1,0,1,0,1)^T$, respectively.

For this example, the expanding point for all methods is selected as $\mathbf{X}^0 = (1.1,1.1,1.1,1.1,1.1,1.1,1.1)^T$, the matching points for TANA-1 and TPA are selected as $\mathbf{X}^1 = (1.0,1.0,1.0,1.0,1.0,1.0,1.0)^T$. Another matching point for TPA is selected as $\mathbf{X}^2 = (0.8,0.8,0.8,0.8,0.8,0.8,0.8)^T$. Under this circumstance, the exponent indices $p_i$ for seven variables for TANA-1 are $4.5, 4.4, 4.5, 0.3, 0.318, -3.5, 2.488$ and $3.813$ from Reference [5]. The remaining constant for TANA-1 is $\varepsilon = -0.0862$.

The relative errors of various methods with different choices of the direction vector $\mathbf{D}$ are plotted in Figures 3(a)–3(c), respectively. With reference to these figures, it is clear that for this highly non-linear example, TPA behaves better than other approximation methods especially when the design variables are changed along the same direction. As shown in Figure 3(a), for $\mathbf{D}_1 = (1,1,1,1,1,1,1)^T$, the relative error of TPA is only 4.86 per cent when $\alpha = 1.5$, and

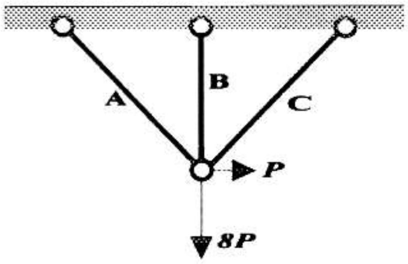


Figure 4. Three bar truss.

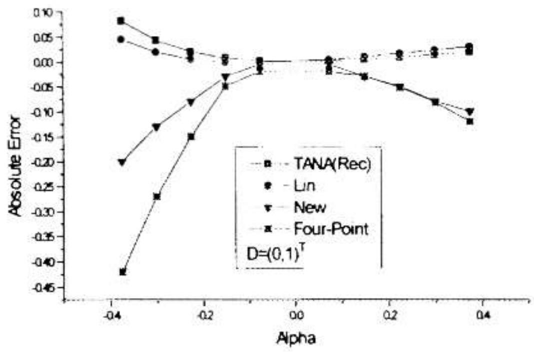


Figure 5. Example 4.

$-10.41$ per cent when $\alpha = -0.8$, whereas TANA-1, linear and reciprocal approach have the error of 110.52, $-77.38$ and $-76.43$ per cent, respectively, when $\alpha = 1.5$, and have the error of $-29.37$, $-93.47$ and $-92.86$ per cent, respectively, when $\alpha = -0.8$. The numerical results obtained here indicate that the proposed approximation scheme TPA has the ability of approximating the highly non-linear function with relatively higher accuracy than the other approximation approaches. For case 3 of this problem, the maximum relative error of TPA approach is $-25$ per cent when $-0.5 \leqslant \alpha \leqslant 0.5$, whereas the TANA-1, linear and reciprocal approximation approach have the maximum error of $-45.64$, $-70.39$ and $-72.51$ per cent, respectively.

Figure 3(b) shows that, when the design variables are changed along alternative opposite direction, TPA also has the best accuracy among all of the compared approximation approaches and improves the accuracy of linear and reciprocal approximation substantially.

### 4.2. *Application to structural optimization*

In this part, the application of the proposed three point approximation approach will be demonstrated by applying it to the solutions of structural optimization problems. Two kinds of truss structural sizing optimization problems have been chosen for the demonstration of the effectiveness of the present method. The constraints include stresses and displacements. The stress constraints are replaced by the equivalent internal force constraints in the implementation of the computer program. The *MOPB optimization tools library* [10] was used to solve the resulting approximate but explicit non-linear optimization problems. For the first three-iterations of structural optimization, a single-point approximation approach (*linear or reciprocal approach*) is used. Then, the process of three-point approximation is used for the subsequent iterations. It is to be noted that in the process of using the three-point approximation approach, the information of the three points is available from the previous iterations and it is not necessary to carry out extra calculations.

*Example 4* (*Three bar truss structure*). The three-bar truss example shown in Figure 4 is taken from Reference [7]. This example has also been used by Wang and Grandhi (1996) to illustrate the effectiveness of the *multi-point Hermite interpolation approach*. The truss is designed for cross-sectional areas $A_A, A_B$ and $A_C(=A_A)$ under stress and displacement constraints. The approximation of member C's stress constraint function is examined. The constraint function using normalized

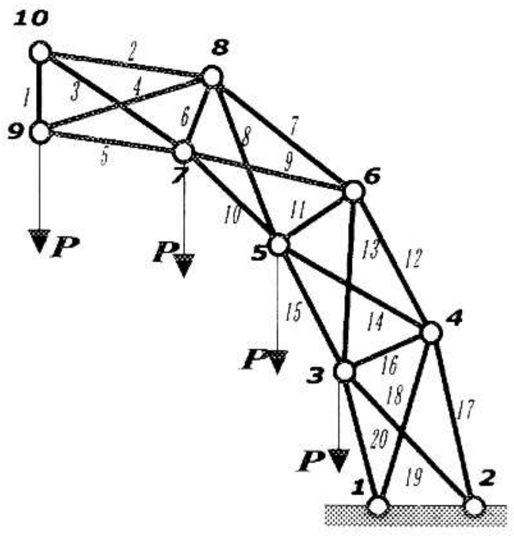


Figure 6. Twenty bar truss structure.

variables can be written as

$$f(x)=1+\frac{\sqrt{3}}{3x_1}-\frac{2}{x_2+0.25x_1} \tag{27}$$

This example is examined by employing the linear, reciprocal, TANA, four-point *Hermite interpolation* and TPA methods with direction vector $\mathbf{D}=(0,1)^{\mathrm{T}}$. The expanding point for all methods is selected as $\mathbf{X}^0=(1.0,1.0)^{\mathrm{T}}$. The matching point for TANA is $\mathbf{X}^1=(1.5,1.5)^{\mathrm{T}}$. The four interpolation points for *Hermite interpolation* approach are taken as $\mathbf{X}^1=(1.5,1.5)^{\mathrm{T}}$, $\mathbf{X}^2=(1.25,0.75)^{\mathrm{T}}$, $\mathbf{X}^3=(1.2,1.2)^{\mathrm{T}}$ and $\mathbf{X}^4=(0.75,1.0)^{\mathrm{T}}$, respectively. The matching points required for TPA are selected as $\mathbf{X}^1=(1.5,1.5)^{\mathrm{T}}$ and $\mathbf{X}^2=(0.8,0.7)^{\mathrm{T}}$, respectively.

The absolute errors of various approximation methods are plotted in Figure 5. It should be noted that TANA approximation approach has the same accuracy as reciprocal approximation approach because its non-linear indices for all design variables are equal to $-1$. Figure 5 also shows that, although it is very difficult to do better than the reciprocal approximation for this problem, the present TPA scheme still behaves better than reciprocal approximation approach when $\alpha\leqslant 0$. The accuracy of TPA is much better than that of linear and four-point *Hermite interpolation* approach.

*Example 5* (*Twenty-bar truss structure*). A 20-bar planar truss shown in Figure 6 is studied in this example. The truss model is taken from Reference [11]. The material properties and nodal loading are given as Young's modulus $E=1.0E+04$, mass density $\rho=0.1$, allowable stresses $\sigma_a=\pm 20$, one load case with $P_{3y}=P_{5y}=P_{7y}=P_{9y}=-100$. The geometry data is listed in Table I.

The internal force of bar-17 was approximated by using linear, reciprocal and the proposed TPA scheme with $\mathbf{D}_1=(10*1,10*-1)^{\mathrm{T}}$, and $\mathbf{D}_2=(2*-1,2*1,2*-1,2*1,-1,+1,4*-1,5*1)^{\mathrm{T}}$, respectively.

Table I. Geometry data for 20-bar structure.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $x$ | 0 | −200 | −76 | −261 | −293 | −435 | −617 | −694 | −1000 | −1000 |
| $y$ | 0 | 0 | 383 | 306 | 607 | 565 | 824 | 739 | 800 | 1000 |

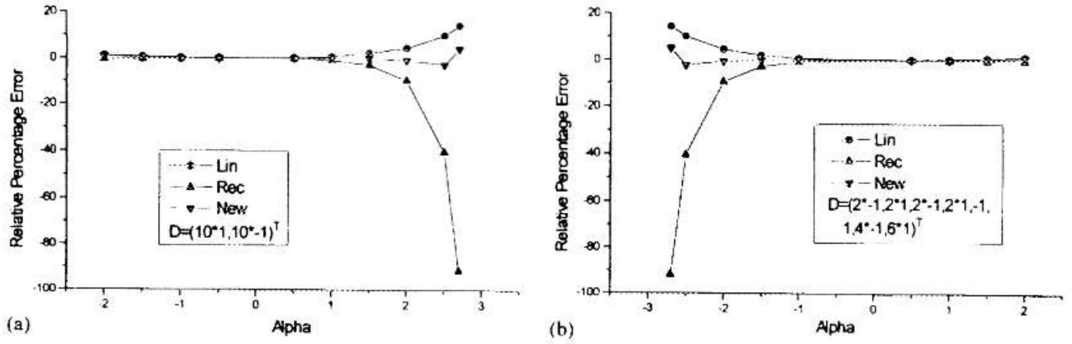


Figure 7. Example 5: (a) case 1; (b) case 2.

Table II. Iteration histories of 20-bar truss with stress constraints.

| Iteration number | TPA weight | Linear weight |
|---|---|---|
| 1 | 32298.3 | 32298.3 |
| 2 | 9504.41 | 9504.41 |
| 3 | 9402.19 | 9402.19 |
| 4 | 9473.52 | 9473.52 |
| 5 | 9430.89 | 9527.77 |
| 6 | 9425.56 | 9504.77 |
| 7 | | 9457.31 |
| 8 | | 9451.72 |
| 9 | | 9426.89 |

The expanding point for all methods are selected as $\mathbf{X}^0 = (6,3,4,6,10,15,5,7,9,10,5,16,7,5,4,10,15,3,5,6)^{\mathrm{T}}$. The other two points required for TPA are selected as $\mathbf{X}^1 = \mathbf{X}^0 + (1,\dots,1)^{\mathrm{T}}$ and $\mathbf{X}^2 = \mathbf{X}^0 - (1,\dots,1)^{\mathrm{T}}$, respectively.

The relative errors of various approximation methods are plotted in Figures 7(a) and 7(b). Both figures show that TPA also works very well for this example. It is better than linear and reciprocal approximation schemes for most of the test points.

In this example, the optimization of the 20-bar truss structure with stress constraints for all bars is also investigated. For this problem, the initial value and minimum size limit are taken as 50.0 and 0.01, respectively, for each design variable. In this example, the linear approximation approach is used for the first three iterations of optimization. Iteration histories of structural weight with 50 per cent move limit are shown in Table II. As Table II shows, TPA approach needs only 6 iterations, whereas linear approximation approach requires 9 iterations to converge.

 

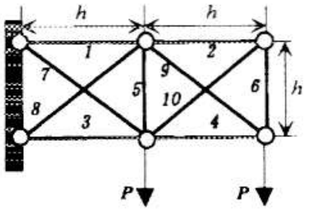


Figure 8. Ten bar truss structure.

Table III. Iteration histories of 10-bar truss with stress constraints.

| Iteration number | TPA weight | Linear weight |
|---|---|---|
| 1 | 4196.49 | 4196.49 |
| 2 | 2071.68 | 2071.68 |
| 3 | 2062.55 | 2062.55 |
| 4 | 2014.45 | 2014.45 |
| 5 | 1992.21 | 1997.84 |
| 6 | 1985.50 | 1988.86 |
| 7 | 1980.90 | 1984.43 |
| 8 | | 1982.22 |
| 9 | | 1981.83 |
| 10 | | 1980.45 |

*Example 6 (Ten-bar truss structure).* A 10-bar truss sizing optimization problem subjected to stress constraints is considered. The 10-bar truss structure is shown in Figure 8. Geometrical and material data are given as Young's modulus $E = 1.0E+04$, mass density $\rho = 0.1$, allowable stresses $\sigma_a = \pm 20$, one load case with $P = -100$. The initial value and minimum size limit are taken as 10.0 and 0.01, respectively, for each design variable. In this example, for the first three iterations of optimization, the linear approximation approach is used. Iteration histories of structural weight with 50 per cent move limit are shown in Table III. The results show that TPA approach has a better efficiency for this optimization problem than the linear approximation approach. Only 7 iterations are required to obtain the optimal solution. It should be pointed out that, the move limit of the optimization process with TPA approximation approach should be reduced carefully during iteration (20 per cent for all of the test examples). With inappropriate move limit, the optimization solution may become unfeasible. The automatic and smart choice of the move limit is still an open research topic.

Next, the same 10-bar truss was optimized under stress and displacement constraints on each vertical degree of freedom. The displacement limit is $\pm 5.0$. Iteration histories of structural weight with 50 per cent move limit are shown in Table IV. In this example, the reciprocal approximation approach is used for the first three iterations of optimization. The results show that TPA approach can also lead to faster convergence to the optimum solution than the reciprocal approximation approach. Only 8 iterations are required to reach the optimum, whereas reciprocal approximation approach requires 13 iterations to converge.

 

Table IV. Iteration histories of 10-bar truss with stress and displacement constraints.

| Iteration number | TPA weight | Reciprocal weight |
|---|---|---|
| 1 | 4196.49 | 4196.49 |
| 2 | 2948.16 | 2948.16 |
| 3 | 2525.50 | 2525.50 |
| 4 | 2435.34 | 2435.34 |
| 5 | 2149.42 | 2381.35 |
| 6 | 2176.59 | 2342.09 |
| 7 | 2239.05 | 2309.64 |
| 8 | 2200.11 | 2284.01 |
| 9 | | 2264.57 |
| 10 | | 2231.62 |
| 11 | | 2220.56 |
| 12 | | 2208.34 |
| 13 | | 2202.31 |

## 5. CONCLUSION

In the present study, a newly developed three-point approximation scheme is proposed for obtaining the high-quality approximation of highly non-linear functions involved in the problem of structural optimization. This scheme is constructed by the linear combination of Taylor expansions in terms of both original and reciprocal variables. The coefficients of the combination are determined by utilizing both the function and gradient information of three different design points obtained during the process of optimization. Based on this approach, the accuracy of the existing constraint approximation methods is improved. The numerical results for the solutions of structural optimization problems indicate that the present method possesses the ability of obtaining the optimum design in less optimization cycles. Thus, the computational efforts associated with the structural re-analyses can be reduced. Another advantage of the present TPA approach is that the unknown coefficients of the approximated function can be obtained in a closed form, and no iterative process is required for the computation of these parameters.

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